Construction of the phase operator using logarithm of the annihilation operator

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Abstract. We investigate a lema that excludes existence of the phase operator and present a condition to avoid the lema. A method for construction of an analytic function f of the annihilation operator a is given. $f(z)$ is analytic on some compact domain that does not separate the complex plane. Using these results we obtain $\ln a$. Since $[a^\dagger a\ , -i \ln a] = i$, we can use $\ln a$ to construct an operator Φ, which satisfies the definition of the phase operator.

PACS numbers: 03.65.Ca, 03.65.Ta

1. Introduction

The annihilation operator in Hilbert space is operator a which satisfies relation:

$$[a, a^\dagger] = 1. \tag{1}$$

A self-adjoint number operator N and the corresponding Fock states are constructed using a:

$$N = a^\dagger a, \qquad N|n\rangle = n|n\rangle, \quad n = 0, 1, 2, 3, ... \tag{2}$$

The phase operator is an operator conjugated to number operator:

$$[N, \Phi] = iI. \tag{3}$$

It is an operator which corresponds to the phase of the harmonic oscillator. Its eigenvalues are phases of the quantum harmonic oscillator. An explicit form of this operator has not been given, and some authors have questioned its existence [1],[2]. Best approach to construction of the phase operator is given by Pegg and Barnett [3]. They constructed an operator and corresponding eigenfunctions which give a good approximation in finite dimensional space. However, when dimension of space tends to ∞, this operator does not exists as an operator on a Hilbert space.

Although form of the phase operator has not been given, basis of its eigenvalues is well known [4]:

$$|\varphi\rangle_f = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} e^{ik\varphi} |n\rangle \ . \tag{4}$$

The vectors $|\varphi\rangle_f$ form a non-orthogonal basis:

$$_f\langle\varphi'|\varphi\rangle_f = \frac{1}{2}\delta(\varphi' - \varphi) + \frac{1}{4\pi}(1 + i\cot\frac{1}{2}(\varphi' - \varphi)). \tag{5}$$

The $|\varphi\rangle_f$ vectors have a useful property, which in fact recommends them as eigenvectors of the phase operator. Their time evolution is natural:

$$\mathrm{e}^{iNt}|\varphi\rangle_f = |\varphi + t\rangle_f. \tag{6}$$

A recent result, giving an expression for analytic functions of the annihilation operator [5], is used here as a starting point for construction of the phase operator. Hence, for the first time we can construct a correct expression for $\ln a$, which will then be used as a base for the further construction of the phase operator. Namely, an explicit construction of an operator which satisfies commutation relation (3) and has vectors (4) as eigenvectors, is given. In section 2, we give a set of conditions for operator Φ which circumvent the non-existence arguments. In section 3, we construct an expression for analytic functions of the annihilation operator and discuss some of its properties. In section 4 we construct $\ln a$. In section 5, starting from $\ln a$, we construct the phase operator Φ.

2. Existence of the phase operator

The usual proof of non existence of the phase operator is based on reduction to contradiction [2]. Namely, assuming this operator exists, using (2) and (3) we obtain:

$$\langle n|[N, \Phi]|m\rangle = (n - m)\langle n|\Phi|m\rangle \neq iI. \tag{7}$$

However, if it is assumed that the vectors $|n\rangle$ do not belong to the domain of the operator Φ, then argument (7) breaks down. We see that a necessary condition for existence of the phase operator Φ is:

$$|n\rangle \notin D(\Phi), \quad n = 0, 1, 2, 3, ... \tag{8}$$

leading to that Φ in $|n\rangle$ representation cannot be expressed as a matrix. If Φ exists, it can be expressed as a product of two or more matrices. Indeed, in section 5, we will construct an operator Φ that satisfies these conditions.

3. Analytic functions of the annihilation operator

In this section, we briefly describe a construction of function f of the annihilation operator a. $f(z)$ is analytic on some compact domain that does not separate the complex plane. A more detailed analysis of this topic is given in [5]. First, we give note about Runge's approximation theorem and a sequence of polynomials which approximate $f(z)$ on the whole domain. Then we construct a new form of identity which is well suited for construction of $f(a)$. We proceed constructing $f(a)$ and finish with an analysis of some of its properties.

3.1. Runge's approximation theorem

Theorem: If f is an analytic function on a compact domain Ω that does not separate the complex plane, then there exists a sequence $P_l(z)$ of polynomials such that converges uniformly to $f(z)$ on Ω [8],[9],[10]:

$$f(z) = \sum_{l=0}^{\infty} P_l(z - z_0) = \sum_{l=0}^{\infty} \sum_{k=0}^{d_l} c_k^{(l)} (z - z_0)^k, c_k^{(l)} \in \mathbb{C}, z, z_0 \in \Omega. \tag{9}$$

Runge's theorem is an existence theorem, i.e. it does not give values of $c_k^{(l)}$. Functions which can be approximated using polynomials (9) are also $\ln z$ and $z^\lambda, \lambda \in \mathbb{R}$. In this paper Mittag-Leffler expansion [9] is used to approximate function $\ln z$.

3.2. New identity resolution

Eigenstates $|\alpha\rangle$ of annihilation operator a are called coherent states:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \alpha \in \mathbb{C}. \tag{10}$$

$|\alpha\rangle$ can be expressed in terms of Fock states $|n\rangle$:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \alpha \in \mathbb{C}. \tag{11}$$

Non-normalized coherent states are:

$$\widetilde{|\alpha\rangle} = \mathrm{e}^{\alpha a^\dagger} |0\rangle = e^{\frac{|\alpha|^2}{2}} |\alpha\rangle. \tag{12}$$

Coherent states form an overcomplete and non-orthogonal set which spans the resolution of identity [7]:

$$I = \frac{1}{\pi} \int \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha|. \tag{13}$$

In formal analogy with the spectral theorem one can write an entire function of the annihilation operator:

$$f(a) = \frac{1}{\pi} \int \mathrm{d}^2\alpha f(\alpha) |\alpha\rangle\langle\alpha|. \tag{14}$$

However, expression (14) is not valid for non-entire functions [7], so we construct a new identity resolution:

$$I = -i \oint_{|\gamma|=R} \frac{\mathrm{d}\gamma}{\gamma} \widetilde{|\gamma + z_0\rangle} \widetilde{\langle\gamma|}\ J\ \mathrm{e}^{-z_0 a^\dagger}\ , R > 0, z_0 \in \mathbb{C}, \tag{15}$$

where

$$J = \frac{1}{2\pi} \sum_{n=0}^{\infty} n! |n\rangle\langle n|. \tag{16}$$

3.3. Construction of analytic functions of the annihilation operator

Using sum (9) and resolution of identity (15), an analytic function of annihilation operator can be constructed:

$$\begin{aligned} f(a) &= \sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)}(a-z_0)^k \cdot I \\ &= -i\sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)} \oint\limits_{|\gamma|=R} \mathrm{d}\gamma\gamma^{k-1}|\widetilde{\gamma+z_0}\rangle\langle\widetilde{\gamma}|\; J\,\mathrm{e}^{-z_0 a^\dagger}. \end{aligned} \tag{17}$$

Using definition (12) of $|\widetilde{\gamma}\rangle$ and performing integration, we obtain

$$f(a) = \sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)} \sum_{n=0}^{\infty}\sum_{m=k}^{n+k} \binom{n}{m-k}\sqrt{\frac{m!}{n!}}\; z_0^{n-m+k}\;\; |n\rangle\langle m|\,\mathrm{e}^{-z_0 a^\dagger}, \tag{18}$$

which for $\alpha \in \Omega$ gives:

$$f(a)|\alpha\rangle = f(\alpha)|\alpha\rangle. \tag{19}$$

We can rewrite (18), collecting coefficients at dyads, as

$$\begin{aligned} f(a) &= \hat{\chi}\mathrm{e}^{-z_0 a^\dagger}, \qquad (20)\\ \hat{\chi} &= \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \chi_{nm}^{(l)}|n\rangle\langle m|\;, \\ \chi_{nm}^{(l)} &= \sqrt{\frac{m!}{n!}}\sum_{k=p}^{s_l} c_k^{(l)}\binom{n}{m-k}\; z_0^{n-m+k}\;, \\ p &= \max\{0, m-n\}, \quad s_l = \min\{m, d_l\}. \end{aligned}$$

3.4. Some properties of analytic functions of annihilation operator

Considering the basic relation between the annihilation and creation operator $[a, a^\dagger] = 1$, one would expect

$$[f(a), a] = 0, \tag{21}$$

$$[f(a), a^\dagger] = f'(a) \tag{22}$$

Since $f(a)$ is not an entire function of a, relations (21) and (22) need to be proven directly. Relation (21) can be calculated explicitly using (20). To prove relation (22) we first need to construct $f'(z)$. It is easy to see that

$$
\begin{aligned}
f'(z) &= \sum_{l=0}^{\infty} \sum_{k=0}^{d_l-1} \bar{c}_k^{(l)} (z-z_0)^k, \ z, z_0 \in \Omega, \\
\bar{c}_k^{(l)} &= (k+1) c_{k+1}^{(l)}.
\end{aligned} \tag{23}
$$

Using (23) and (20) we can prove (22) directly.

4. Logarithm of the annihilation operator

In this section we construct an operator $\ln a$, which is a good starting point for construction of the phase operator [6],[7]. A complex function $f(z) = \ln z$ is analytic on a simply connected subdomain Ω in $\mathbb{C}$, which is obtained by making a cut in the complex plain along a ray originating at zero. For simplicity, we chose a cut along the negative part of the x axis. We also set $z_0 = 1$ in (9). Any other choice of a cut and z_0 leads to an equivalent construction of $\ln a$. As already noted, the Runge theorem does not explicitly give coefficients $c_k^{(l)}$ in (9). A convenient method for computing $c_k^{(l)}$ for the function $f(z) = \ln z$ is Mittag-Leffler expansion in the star [9]. This method gives explicit constants in the expansion of $\ln z$:

$$
\begin{aligned}
\ln z &= \lim_{p\to\infty} \sum_{l=1}^{p} \sum_{k=1}^{p} \ c_k^{(l)} \ (z-1)^k \ , \ \ z \in \Omega, \\
c_k^{(l)} &= d_k^{(l)} \ \frac{(-1)^{(k+1)}}{k}, k > 0, \\
d_k^{(l)} &= \frac{k!}{l!} \ \Gamma_p^k \ \Theta_p^l \ E_l^{(k)} \ ,
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
\Gamma_p &= 2H_p^{-2}, \\
\Theta_p &= 1 - e^{-\frac{1}{2}H_p^2}, \\
E_l^{(k)} &= \frac{1}{k!} \frac{\partial^k}{\partial \rho^k} \prod_{s=0}^{l-1} (\rho + s) \ \Big|_{\rho=0} = c(l,k).
\end{aligned} \tag{25}
$$

c(l,k) is unsigned Stirling number of the first kind. H_p is any sequence satisfying the following condition:

$$
2H_p^2 \ e^{\frac{1}{2}H_p^2} < p, \qquad p > 0. \tag{26}
$$

This sequence of polynomials converges locally uniformly to $\ln z$ on Ω. Using the obtained coefficients in the expansion (24) and relation (20) we can represent $\ln a$ as the following limit:

$$
\ln a = \lim_{p\to\infty} \sum_{l=0}^{p} \sum_{n=1}^{p} \sum_{m=1}^{p} \chi_{nm}^{(l)} |n\rangle\langle m| \ \ \mathrm{e}^{-a^\dagger}, \tag{27}
$$

$$\chi_{nm}^{(l)} = \sqrt{\frac{m!}{n!}} \sum_{k=k1}^{m} c_k^{(l)} \binom{n}{m-k} \quad , k1 = \max\{0, m-n\}.$$

$\ln a$ has nice properties needed for construction of the phase operator [7]. Using (21) and (22) it is obvious that:

$$[a^\dagger a, -i\ln a] = a^\dagger[a, -i\ln a] + [a^\dagger, -i\ln a]a = \frac{i}{a}\ a = i. \tag{28}$$

We can conclude that $\ln a$ is conjugate to number operator, and therefore is a good base for construction of the phase operator.

5. $\ln a$ and the phase operator

Let

$$\Phi = -iY^{-1}\ln aY, \tag{29}$$

where

$$Y = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} |k\rangle\langle k|. \tag{30}$$

Y is a diagonal operator so it is obvious

$$[a^\dagger a\ , \Phi] = i. \tag{31}$$

On the other hand, vectors (4) and coherent states (10) can be combined:

$$|\varphi\rangle_f = \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} Y^{-1} |e^{i\varphi}\rangle \tag{32}$$

Vector $|\varphi\rangle_f$ is an eigenvector of the operator Φ in equation (29):

$$\Phi|\varphi\rangle_f = -iY^{-1}\ln aY \frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} Y^{-1}|e^{i\varphi}\rangle = -i\frac{e^{\frac{1}{2}}}{\sqrt{2\pi}} Y^{-1}(i\varphi)|e^{i\varphi}\rangle = \varphi|\varphi\rangle_f, \tag{33}$$

$$\varphi \in (-\pi, \pi).$$

Hence, the operator Φ is a phase operator.

6. Appendix

To determine action of the operator $f(a)$ defined in (18), on the vector $|\alpha\rangle$, we first compute the following matrix element:

$$\langle m|e^{-z_0 a^\dagger}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \langle m||\widetilde{\alpha - z_0}\rangle = e^{-\frac{|\alpha|^2}{2}} \frac{(\alpha - z_0)^m}{\sqrt{m!}} \tag{34}$$

Using (18) we can see

$$f(a)|\alpha\rangle = \sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)} \sum_{n=0}^{\infty}\sum_{m=k}^{n+k} \binom{n}{m-k} \frac{1}{\sqrt{n!}} \; z_0^{n-m+k} e^{-\frac{|\alpha|^2}{2}} (\alpha - z_0)^m \; |n\rangle$$
$$= e^{-\frac{|\alpha|^2}{2}} \sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)} (\alpha - z_0)^k \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{m=0}^{n} \binom{n}{m} z_0^{n-m} (\alpha - z_0)^m \; |n\rangle$$

Last sum in previous equation is α^n, and finally

$$f(a)|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{l=0}^{\infty}\sum_{k=0}^{d_l} c_k^{(l)} (\alpha - z_0)^k \sum_{n=0}^{\infty} \frac{(\alpha - z_0)^n}{\sqrt{n!}} \; |n\rangle = f(\alpha)|\alpha\rangle \qquad (35)$$

Acknowledgments

The autor would like to thank Drs M. Arsenović, D. Arsenović, D. Davidović and J.Ajtić for their suggestions and help in preparation of this manuscript.

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